THE AC RESISTANCE AND INDUCTANCE OF RAILS

To detect the presence of a train on a railway track an AC voltage is applied to the rails and this is shorted-out by the train. The effectiveness of this system depends on the resistance and inductance of the rails but quantifying these is difficult because the rails are made from magnetic steel whose permeability and loss change with frequency, and with the amplitude of the current. A further difficulty is the complicated shape of the rail cross-section. This report models the rail as having a rectangular shape and then derives equations for the resistance and inductance and these agree well with published measurements.

1. INTRODUCTION

1.1. The Track Circuit

To determine whether a section of railway track has a train on it, the two rails are held at different electrical potential and are connected together by the wheels and axle of locomotives and rolling stock which then short out an electrical circuit. This circuit is monitored by electrical equipment to detect the absence or presence of the trains, and the whole is known as a Track Circuit. A simple DC circuit is shown below using a battery and a relay to signal the presence of the train:

![Simple DC track circuit](image)

Figure 1.1 Simple DC track circuit

However, DC track circuits cannot be used where DC traction is used on the running line or on tracks in close proximity. Similarly if 50 Hz AC electrification is used then 50 Hz AC track circuits cannot be used and then the AC frequency used is in the range of audio frequencies, from 91 Hz up to 10 kHz.

The effectiveness of this system depends greatly upon the resistance and inductance of the rails at audio frequencies, but determining these theoretically is complicated by the fact that the rails are made from magnetic steel whose magnetic permeability and magnetic loss vary with frequency and with the magnitude of the current. The audio current will be small at around 160 mA and this will have negligible effect on the rail characteristics but there can be a large traction current of 200-1900 A.

In addition to these problems, at high frequencies the current is carried in a thin skin around the periphery of the rail (the skin effect), but computing this is difficult because of the complicated cross-section of a rail shown below.
Figure 1.2 Parts of a modern rail

The above shape, a ‘flat footed rail’ is now used on 90% of the world’s railways and is sometimes called a Vignole rail after Charles Vignole who invented it in 1836. Rails are made of high carbon steel and to increase the service life the head is sometimes hardened to a depth of about 12 mm from the surface. In the context of this report it is important to note that the steel is chosen for its mechanical properties only, and the electrical characteristics of the steel are seldom part of the specification. A brief survey of the literature shows a wide range of possible electrical values (Appendix 1).

1.2. Theoretical Analysis

In many of the published papers the track impedance is analysed using transmission-line simulation software, but often this does not model the necessary complexity of rail and track-bed. While it would be possible to improve this software transmission-line theory is not essential here because the length of the track over which the signals are to be transmitted is generally very much less than the wavelength of the signal. For instance at 10 kHz the wavelength ($\lambda_T$) along the track is about 7 km (ref 27) and main-line track circuits are generally no longer than 1 km so at 10 kHz the longest track circuit will be about 0.14 $\lambda_T$ and lumped impedance analysis is then normally sufficiently accurate. A comparison between the analytical approach used here and the approach used by many authors is given in Section 7. The analysis here uses a lumped element model for the rails consisting of a series resistance and series inductance. The apparent simplicity of this circuit hides the incorporation of complicated frequency dependence of the values of these components, and it is this complexity which is the main focus of this report.

The main topic of this paper is the rail resistance and inductance but the effects of the rail to rail impedance (ie through the track-bed) are also included because the measurements of the rail parameters on a real track cannot be made in isolation (Section 5). Here the track-bed impedance is modeled as a network rather than the simple parallel RC conventionally used, and this is shown to be a more accurate representation.

A major problem with measurements conducted on real tracks is that the magnetic characteristics of the rails are seldom known, but in these cases it has been found that realistic magnetic parameters can be selected in the theoretical model to give a good correlation with the measurements.

In this article the most significant equations are given in red.

2. THE MAGNETIC PERMEABILITY OF RAILS

Materials with a high permeability owe their magnetic properties to the ability of the material to organize itself into magnetic domains. In an un-magnetised material these domains are oriented randomly and so the net magnetisation is zero. When a small external field is introduced some domains move at the expense of others, increasing the overall magnetization and his can give the material a permeability of many thousands. The domains have mass and so at high frequencies the movement is reduced in amplitude, and has associated with it frictional energy loss.

So the material permeability has a real component $\mu'$ and an imaginary component $\mu''$ and the overall permeability is given by :
\[ \mu_m = \mu' - j \mu'' \]  \hspace{1cm} \text{2.1}

Notice that the convention here is for the reactive component \( \mu' \) to be real and the loss component \( \mu'' \) to be imaginary, the opposite to the normal circuit representation. The ratio of these two components give the Q of the material, and this is then equal to ratio of the rail reactance to the rail resistance:

\[ Q_m = \frac{\mu'}{\mu''} = \frac{\omega L_I}{R_M} \]  \hspace{1cm} \text{2.2}

So the series resistance due to the magnetic loss in the rail is equal to:

\[ R_M = \frac{\omega L_I \mu''}{\mu'} \]  \hspace{1cm} \text{2.3}

where \( L_I \) is the internal inductance of the rail (see later)

Both \( \mu' \) and \( \mu'' \) vary with frequency and the graph below shows these characteristics:

![Permeability vs Frequency Graph](image)

**Figure 2.1 Permeability \( \mu_m' \) and \( \mu_m'' \)**

In the above example the real component (blue curve) has an initial permeability \( \mu_I' \) of 300, and a relaxation frequency \( f_m \) of 1 KHz. The equation which describes this characteristic is:

\[ \mu' = (\mu_I' - 1) / [1 + (\mu / f_m)^2] + 1 \]  \hspace{1cm} \text{2.4}

where \( \mu_I' \) is the initial (low frequency) permeability

\( f_m \) is the magnetic relaxation frequency of the material

The loss component shows a resonant characteristic with an amplitude equal to that of the real component at the frequency \( f_m \). Its equation is:

\[ \mu'' = \mu_I' \left( \frac{\mu}{f_m} \right) / [1 + \left( \frac{\mu}{f_m} \right)^2] \]  \hspace{1cm} \text{2.5}

At high frequencies Equation 2.4 is asymptotic to unity, and this is typical of many magnetic materials especially ferrites. However Bowler (ref 2) has measured a higher asymptote for steel and a different rate of roll-off so the equations need to be modified to:

\[ \mu' \approx (\mu_I' - \mu_{\infty}) / [1 + \left( \frac{\mu}{f_m} \right)^n] + \mu_{\infty} \]  \hspace{1cm} \text{2.6}

where \( \mu_{\infty} \) is the high frequency permeability

As an example Bowler measured samples of 1018 low-carbon steel and found \( \mu_I' \approx 250, \mu_{\infty} \approx 80 \) and \( f_m \approx 5000 \) Hz.

It is not clear how to modify Equation 2.5, but it appears that at high audio frequencies the ratio \( \mu' / \mu'' \) is about 3.5 (ref 29). So it is assumed that Equation 2.5 becomes:
\[ \mu'' \approx (\mu_1'' - \mu_\infty''/3.5) (f/f_m) / [1+ (f/f_m)^n] + \mu_\infty'' /3.5 \]  

\[ 2.7 \]

3. THE RESISTANCE OF RAILS

3.1. Equivalent Rectangular Conductor

At high frequencies the current tends to concentrate around the periphery of the rail. This is known as the skin effect and this considerably increases the resistance of the rails. The complicated rail cross-section does not lend itself to accurate analysis of the skin effect, and many authors have substituted a circular conductor with the same periphery as that of the rail. This can give reasonable results at high frequencies but at DC and low frequencies conduction takes place throughout the whole rail cross-section and it is then the cross-sectional area which is important. The circular conductor will have a different area to that of the rail and so fail to model the resistance accurately. Of course the circular conductor can be chosen to have the same area as the rail but then its periphery will be different.

A rectangular conductor provides a much better model for a rail, since it can simultaneously have the same area and same periphery as the rail, and also has current crowding at its corners which is not present of course in a circular conductor. However the AC resistance of a rectangular conductor is much more difficult to analyse than the circular conductor, but recently the author has developed a simple semi-empirical equation which is surprisingly accurate (ref 9). This is given later.

For the rectangle to have the same cross-sectional area \( A \) as the rail and the same periphery \( p \), its thickness and its width need to be (see Appendix 2):

\[
\text{Thickness } t = \frac{p}{2} - \sqrt{\left(\frac{p}{2}\right)^2 - 4A} / 2 \]

\[
\text{Width } w = \frac{p}{2} - t \]

3.1.1

The above uses the normal nomenclature for a rectangular bar where its widest dimension is the width \( w \) and its narrowest dimension is the thickness \( t \).

For instance for the R65 rail \( w = 314 \) mm and \( t = 26.4 \) mm, and for the UIC 60 rail \( w = 317 \) mm and \( t = 24 \) mm. This rectangular model is illustrated below along with the circular model.

Figure 3.1.1 Rectangular and circular model
3.2. Skin Depth
At DC the current will flow in the whole cross-section of the rail, but at high frequencies it will concentrate in a skin around the periphery of the rail with an effective thickness given by:

$$\delta = \left[ \frac{\rho}{(\pi f \mu_\mu_r)} \right]^{0.5} \text{ metres} \quad 3.2.1$$

Where $\mu_0 = 4 \pi \times 10^{-7} \text{ H/m}$
$\mu_r$ is the relative magnetic permeability of the steel rail
$f$ is the frequency in Hz
$\rho$ is the resistivity of the steel rail in Ohm meters

The skin depth determines the resistance of rails over most of the useful frequency range as shown by the following graph of skin depth for $\rho = 0.25 \times 10^{-6}$ (an average value – see Appendix 1) and $\mu_r = 60$:

![Skin Depth vs Frequency Graph](image)

Figure 3.2.1 Skin depth in rail

So at 100 Hz the current is carried in a thin skin only 3.5 mm thick around the periphery of the rail, and this reduces to 1 mm or less at higher frequencies. The resistance over most of the frequency range is therefore approximately equal to the DC resistance increased by the ratio of the total cross-sectional area to that of the skin. Also skin-effect reduces the inductance in that the penetration of magnetic flux into the conductor is also limited to the same depth of skin.

3.3. Rail Resistance
The author has recently derived the following equation for the resistance of a rectangular conductor (ref 9):

$$R_{ac} \approx R_{dc} K_C / (1 - e^{-x}) \quad \text{Ohms} \quad 3.3.1$$

The term $K_C$ describes the current crowding which occurs at the edges and corners, and $(1 - e^{-x})$ describes the diffusion of current into the conductor, the skin effect.
To this must be added the resistance due to the magnetic losses, Equation 2.3 (the effect of the track-bed is included in Section 5). The total resistance of a section of track of length $\ell_T$, shorted at the far end is then:

$$R_{ac} \approx R_{dc} K_C / (1 - e^{-x}) + \omega L_T \mu'' / \mu' \quad \text{Ohms} \quad 3.3.2$$

Where $K_C = 1 + F(0)$
$F(0) = (1 - e^{-0.026 \rho})$
$\rho = A^{0.5} / (1.26 \delta)$
$A$ is the cross-sectional area of the conductor
$w$ and $t$ are given by Equation 3.1.1
$\delta$ is the skin depth (Equation 3.2.1)
$x = 2(1+t/w) \delta/t$
R_{dc} = [\rho 2l_T / (w t)]

l_T is the track length
\rho is the rail resistivity in Ohm meters
L_I is the internal inductance of the rail (see later)
\mu' is given by Equation 2.6
\mu'' is given by Equation 2.7

NB The equation for x is not the final equation given in ref 9, but that above is simpler and is sufficiently accurate for the w/t ratio here. Also for the same reason the equation for K_C has been simplified.

In the above it is assumed that the resistivity is constant with frequency, and there is no evidence to the contrary.

3.4. Other Contributors to Resistance
The following contributors to the resistance were considered but their effects were found to be not significant.

3.4.1. Proximity loss
The current flowing through a rail will generate a magnetic field around it and this will encompass the other rail and induce a current in it (and indeed any other conductor in the vicinity). The power loss from this induced current must come from the first rail and so its resistance increases, and this increase is given for a circular conductor by (ref 6):

\[ \frac{R}{R_0} = 1/ \left[ 1 - \left( \frac{d}{D} \right)^2 \right]^{0.5} \]

where \( d \) is the diameter of the conductor
\( D \) is the centre to centre separation

No equation has been found for a rectangular conductor and so it is assumed here that the diameter \( d \) is equal to the width \( w \), and this is likely to be a good approximation at large spacings. So:

\[ \frac{R}{R_0} \approx 1/ \left[ 1 - \left( \frac{w}{D} \right)^2 \right]^{0.5} \]

for \( D \gg w \)

For the usual track structure where \( w \approx 0.32 \) meters and \( D \approx 1.6 \) meters this gives an increase of only 2% and so the complexity of a more accurate estimate is not justified.

3.4.2. Current Recession
The effective perimeter is slightly smaller than the physical perimeter because the current recedes from the surface of the conductor by half the skin depth (Wheeler ref 7). So a rectangular conductor has an effective width \( w' = w - \delta \), and an effective thickness \( t' = t - \delta \), giving the ratio of the receded periphery to the physical periphery as:

\[ P_{eff} / p = 2[(w - \delta) + (t - \delta)] / [2(w+t)] = 1 - 2 \delta/(w+t) \]

For the R65 rail \( w+t= 340 \) mm. Assuming that the permeability is 27 and the resistivity is \( 0.7 \times 10^{-6} \) then the skin depth \( \delta \) at 200 Hz will be approximately 6mm, and the effective perimeter will be reduced by 3.5%. So the effect is generally much smaller than the uncertainties in the rail parameters and can therefore be discounted.

3.4.3. Radiation Resistance
There will be electromagnetic radiation from the rails and if these were in free space the added resistance due to this, the radiation resistance, would be (ref 18, p167):

\[ R_R = 31200 \ (l_T D)^2 / \lambda^4 \]
For $\ell_T = 2000$ m, $D = 1.6$ m, and $f = 10$ KHz ($\lambda = 3 \times 10^4$ m), then $R_R = 40 \mu\Omega$. This is about 0.1% of the track resistance and can be discounted therefore.

### 4. THE INDUCTANCE OF SHORTED TRACK

#### 4.1. Introduction

The total inductance of the shorted rails is the sum of their external inductance and internal inductance:

$$L_T = L_E + L_I$$  \hspace{1cm} 4.1.1

The external inductance is due to the magnetic field external to the rails and between them, and is independent of frequency. The internal inductance is due to the field within the conductor. This internal field is quite small in non-magnetic materials such as copper and contributes only a small amount to the total inductance as illustrated below (for a circular conductor):

![Figure 4.1.1 Fields for $\mu_R = 1$](image)

With steel conductors the internal field is much larger because of the high permeability as shown below:

![Figure 4.1.2 Fields for $\mu_R >> 1$](image)

The internal inductance reduces with frequency because the penetration of flux into the rails reduces due to the skin effect. At low frequencies the internal inductance can be much greater than the external inductance (as an example see Figure 6.1.2).
4.2. External Inductance of two Shorted Rails

The rails, along with the train wheels which short them together form a return circuit of two conductors. Published equations for the external inductance for this arrangement are for conductors which are circular, rectangular or elliptical. The circular equivalent is often assumed but Appendix 3 shows that this can give a large error. The alternative is the rectangular equivalent and the inductance for this for a non-magnetic conductor is (Terman ref 5 p52):

\[ L_E = \mu_o \mu_r \ell_T / \left( \frac{2\pi}{\ln D/(w+t) + 1.5} \right) \] 4.2.1

Appendix 3 shows that the following equation gives a better approximation to practical rails.

\[ L_E = \mu_{RT} \mu_o \mu_r \ell_T / \left( \frac{2\pi}{\ln D/(w+t) + 1.7} \right) \] 4.2.2

In the above equation \( \ell_T \) is the length of the shorted track and the rail spacing \( D \) is the distance between the centre of the rails. This is wider than the track width which is defined to the inside edges of the head so \( D \) will be equal to the track width plus the head width which for the R65 rail and 1520mm gauge gives a centre to centre distance of 1595 mm.

The relative permeability of the track-bed, \( \mu_{RT} \), is approximately equal to 1.15 (see Appendix3).

4.3. Internal Inductance

Most conductors have a relative permeability of unity and so the inductance due to the internal field is small at around 0.5%. Generally the internal inductance is not included as a separate term in the equations for inductance but if it is, it is usually assumed that the frequency is low enough that there is complete diffusion into the conductor, and thus skin effect can be discounted. For instance the self-inductance of a straight circular conductor of length \( \ell \) is often given as:

\[ L = \mu_o \ell \left( \frac{2\pi}{\ln 4 \ell / d - 1 + \mu_R / 4} \right) \] 4.3.1

where \( \mu_R \) is the low frequency relative permeability of the conductor.

In the above the term \( \mu_o \ell / (2\pi) \mu_R / 4 \) is the internal inductance and notice that the effective permeability is \( 1/4 \) of the relative permeability (see Harnwell ref 11 p330 for the derivation). This reduction in effective permeability arises because at low frequencies the field decreases linearly towards the centre of the conductor as shown in Figures 4.1.1 and 4.1.2. The magnetic field within a rectangular conductor will not be circular and so we can expect a different value which we can designate as \( \mu_R / k_p \). For the value of \( k_p \) the only information found in the literature is from Antonini et al (ref 10) who used numerical methods to analyse the rectangular conductor and they give an example with a w/t ratio of 10.7 and their results correspond to \( k_p = 6.3 \). No measurements have been found, and so the author has carried-out two measurements and these are described in Appendix 4, with the results shown below in red:

Figure 4.3.1 DC Permeability Factor \( k_p \)
Also shown is Antonini’s data in blue, and the known value for the circular conductor of 4, arbitrarily plotted at w/t=0. All points lie close to a straight line given by:

\[ k_P \approx 4 + 0.27 \frac{w}{t} \]  

For the UIC 60 and R65 rails w/t is equal to 11.9 and so \( k_P = 7.2 \), and this is the value used in the comparisons with published measurements Section 6.

At high frequencies the decrease in effective permeability is greater because of the reduced penetration of flux caused by skin effect. The change for a conductor with a circular cross-section is easiest to calculate (ref 4), but even here it was necessary to resort to approximations to avoid Bessel functions since these are difficult to handle. The rectangular cross-section gives even greater difficulties because the current density is not so easily described and there is no known closed solution. However at high frequencies, where the skin depth is much smaller than the cross-sectional dimensions, the resistance of the conductor then increases as \( \sqrt{f} \) and the inductance as \( 1/\sqrt{f} \). The approach here is to use this reciprocal relationship between the resistance and the inductance, so that as the frequency is raised the internal inductance reduces from its DC value to a lower value, assumed to be inversely as the resistance. The change in permeability with frequency must also be taken into account and so \( \mu_R \) is replaced by \( \mu' \) (Equation 2.6).

So from Equation 4.3.1 the internal inductance of a length of track \( \ell_T \) formed with a rectangular conductor is assumed to be given by:

\[ L_I \approx \mu_o \frac{2\ell_T}{(2\pi)} \left( \mu' / k_P \right) \left( 1 - e^{-\frac{x}{K_C}} \right) \text{ Henrys} \]  

4.4. Total Inductance

The total inductance is the sum of the external and internal inductances, Equations 4.2.2 and 4.3.2, so the total inductance of a section of track of length \( \ell_T \) and shorted at the far end is:

\[ L \approx \mu_{RT} \mu_o \frac{2\ell_T}{(2\pi)} \left[ \ln \frac{D}{(w+t)} + 1.6 + \left( \mu' / k_P \right) \left( 1 - e^{-\frac{x}{K_C}} \right) \right] \text{ Henrys} \]  

where

- \( \ell_T \) is the length of the track
- \( D \) is the centre to centre dimension of the track
- \( w \) and \( t \) are defined by Equation 3.1.1
- \( k_P \approx 4 + 0.19 \frac{w}{t} \)
- \( \mu \) is given by Equation 2.6
- \( \mu_o = 4\pi \times 10^{-7} \text{ H/m} \)
- \( \mu_{RT} \) is the track-bed permeability \( \approx 1.15 \) (see Appendix 3)
- \( x \) and \( K_C \) are defined in Equation 3.3.2

The inductive reactance of the short-circuited track is:

\[ X_L = 2\pi f L \text{ Ohms} \]  

where

- \( f \) is in Hz
- \( L \) is given by Equation 4.4.1
5. INCLUSION OF TRACK-BED IMPEDANCE

5.1. Equivalent Circuit of Rails plus Trackbed

The above equations give the rail resistance and reactance of a shorted section of track. In practice these cannot be measured directly because there is an additional impedance between the rails due to currents flowing in the track bed. The equivalent circuit is shown below:

![Equivalent Circuit of rails and track-bed](image)

The rail resistance is $R_R$ and its inductance is $L_R$. In parallel with these is the track impedance $R_S$ and $C_S$, and the value of these is found by measuring the open circuited impedance of the track i.e. with the short removed, as $R_S + jX_{CS}$. The values of $R_S$ and $C_S$ are not constant with frequency and this is considered in Section 5.2.

The total impedance of the shorted track is equal to the input impedance of the above circuit and this is:

$$Z_{IN} = \frac{(R_L + jX_L)(R_S - jX_S)}{(R_L + jX_L + R_S - jX_S)}$$  

5.1.1

When the track is shorted at the far end the effect of the track-bed impedance is considerably reduced: this can be easily appreciated since toward the shorted end of the track the rail-to-rail impedance is shorted-out. Analysis shows that the effective impedance is increased by a factor of 3 assuming a linear voltage gradient along the length of the track (Appendix 9). So in applying the measured open circuit values of $R_C$ and $X_C$ into the above equation they must be multiplied by 3:

$$Z_{IN} = \frac{(R_L + jX_L)(3R_S - j3X_S)}{(R_L + jX_L + 3R_S - j3X_S)}$$  

5.1.2

Assuming $R_S >> X_{CS}$

$$Z_{IN} \approx \frac{[3R_S R_L + jX_L 3R_S]}{[(R_L + 3R_S) + jX_L]}$$

(NB The assumption that $R_S >> X_{CS}$ will not be true at low frequencies, but then the rail inductance and resistance will be low and so the effect of $R_S$ and $X_{CS}$ is small)

Using the identity $(A+jB)/(C+JD) = (AC + jBC - jAD + BD) / (C^2 + D^2)$

$$Z_{IN} = [(3R_S R_L)R_L + 3R_S + 3R_S X_L^2] + j[3R_S X_L (R_L + 3R_S) - 3R_L R_S X_L] / [(R_L + 3R_S)^2 + X_L^2]$$  

5.1.3

where $R_S$ is the measured open circuit track-bed resistance

$R_L$ is the resistance of the rails (Equation 3.3.2)

$X_L$ is the reactance of the rails (Equation 4.4.2)
The above equation assumes that it has been possible to measure the rail impedance and the track-bed impedance independently. In practical measurements on real track the rail impedance (the short-circuited measurement) will be modified by the track-bed impedance and vice versa. The errors which this gives when $Z_{IN}$ is calculated by the above equation are given in Section 8.

5.2. Model of Track-bed

Song (ref 26) gives the following model of the rail to rail impedance, based upon the porous nature of concrete sleepers (see also Appendix 8):

$$\text{Figure 5.2.1 Equivalent Circuit of Trackbed}$$

Measurements on an open track will be in series form $R_S + jX_{CS}$ (Figure 5.1.1) and the relationship to the above circuit is:

$$R_S = R_E + R \cdot \frac{X_C^2}{(R^2 + X_C^2)}$$

$$jX_{CS} = -j \frac{(X_C R^2)}{(R^2 + X_C^2)}$$

5.2.1

This model is consistent with the measurements by Ivanek et al (ref 14) who give the measured open circuit impedance for their 540 metre track as:

$$\text{Figure 5.2.2 Measured open-circuit } R_S \text{ and } X_{CS} \text{ (from Ivanek)}$$

These measurements are modelled by the above equation if $R_e = 13 \Omega$, $R = 10\Omega$, $C = 1900 \mu F$, and this is shown below:
The agreement is good except for the two highest frequency points where the impedance goes from capacitive to inductive. This is most unlikely and points to experimental error, possibly a phase error. The resistance values are very low and the capacitance very high and this is consistent with the statement by the authors that ‘The railway yard was very dirty from coal dust, loam and clay, dry fallen leaves, twigs and so forth. It was raining slightly when the measurements were taken, and had been raining for a long time before-hand’.

For a 1km track these values become \( R_e = 7 \, \Omega \), \( R = 5.4 \, \Omega \), \( C = 3520 \, \mu F \)

More information on track-bed impedance is given in Appendix 8.

### 6. COMPARISON WITH PUBLISHED MEASUREMENTS

#### 6.1. Measurements by Havryliuk and Meleshko

To evaluate the accuracy of the theoretical analysis it is necessary to have experimental measurements and some useful data is given in reference 1 for audio frequencies. This is summarized in their Table 1 as ‘Traction rail impedance for a distance of 1km’, and gives the measured series resistance and series reactance (always inductive) for frequencies between 25Hz and 5555 Hz. These measurements come from another report but unfortunately this is not readily accessible because it is written in Ukranian. Given that the above is for 1km of track, this implies two parallel rails each of 1 km in length.

Unfortunately the open circuit impedance (ie track-bed impedance) is not given, and so it is assumed here that the measured shorted impedance was that of the rails alone.

The authors give the measured resistance in blue below:

![Figure 6.1.1 Measured Resistance and Havryliuk’s analysis](image-url)
They comment that ‘….the data looks like a zigzag broken line that is probably due to errors in the measurements’. Their own analysis is shown dotted above and at 5 kHz their prediction is almost twice the measured value. While clearly there will be some measurement error it is difficult to believe that it could be so great, especially since it can be explained by a change in permeability, as shown below.

Equation 3.3.2 was programmed into excel. The rail was R65 and this has a cross-sectional area of A = 82.65 cm², and a periphery p of 680mm (found by placing string around a drawing of the rail), and so has the rectangular equivalent of w= 314 mm and t=26.6 (Equation 3.1.1). The resistivity ρ is given by the authors as 0.21 10⁻⁶ Ωm and the relative permeability µr = 100, but these are probably nominal values since there is no evidence that these were measured values. Better agreement with the theory here is given with the average value of resistivity of ρ=0.25 10⁻⁶ Ωm (see Appendix 1). The measured track resistance and the theoretical prediction are then shown below:

![Resistance of 1Km of Track](image1)

**Figure 6.1.1 Comparison Theory and Measurements**

The magnetic parameters were optimized for best agreement with the measurements and this gave µ₁ = 51, µ∞ = 7.5, fₘ = 350Hz and n=2 (Equation 2.6). With these assumptions it is seen that the correlation is good over the whole frequency range. Also shown is the resistance contributed by the magnetic loss (Equation 2.3), and this peaks at around 1kHz and is almost half the total resistance. It should be noted that it would not be possible to explain the complicated shape of the measured resistance curve without the frequency dependency of the magnetic permeability, and this is missing in most papers on this subject including that of Havryliuk et al.

The inductance was derived from the measured reactance values (ie L= Xₐ/(2πf)) and this is plotted below. Also shown is the modeled inductance Equation 4.4.1 using the same magnetic parameters as above:

![Inductance of 1 Km of Track](image2)

**Figure 6.1.2 Comparison Theory and Measurements for Inductance**
The correlation is very good at all frequencies, and this is partly because the magnetic permeability of the track $\mu_{\text{RT}}$ (Equation 4.4.1) has been chosen to give the best agreement with the measurements. However the optimum value for the Ivanek measurements was also $\mu_{\text{RT}} = 1.15$ (see Section 6.2), and so this may indicate a generally characteristic of track-bed (see also ref 28). The theoretical curve is extended to low frequencies in the above graph to show the high inductance at these frequencies, caused by the deeper penetration of the magnetic flux into the rail at the low frequencies. At 50 Hz the inductance is 2.25 $\mu$H per metre of track. In practice it is the reactance which is needed and a plot of this shown below (Equation 4.4.2):

![Reactance of 1Km of Track](image)

*Figure 6.1.3 Comparison Theory and Measurements for Reactance*

### 6.2. Measurements by Ivanek et al

#### 6.2.1. Introduction

Ivanek et al (ref 14) measured the input impedance between two rails when they were short circuited at the far end and when they were open circuit. Their measurement technique was to inject a sine-wave current into the rails and to measure the input voltage and current on a twin trace oscilloscope. No details are given on how the phase between these two was determined or of any calibration or likely accuracy. However the authors say that the data ‘contained a large amount of noise…..’.

The length of the track was 540 metres. No other details are given of the rails except that ‘measurements were performed in the Orlova locality in the rail yard of a private firm, AWT. The section measured did not have any track crossings or track branches and had only a slight radius of curvature. The railway yard was very dirty from coal dust, loam and clay, dry fallen leaves, twigs and so forth. It was raining slightly when the measurements were taken, and had been raining for a long time before-hand’. In the absence of any other information it is assumed here that the rails are UIC 60.

#### 6.2.2. Track-bed Resistance

To test Equations 3.3.2 and 4.4.1 against Ivanek’s short circuited measurements the rail to rail impedance across the track-bed must be included (Equation 5.1.3). The authors give this open circuit impedance in terms of series components $R_c + jX_c$ as shown in Figure 5.2.2.
6.2.3. Comparison of Theory and Measurements

No information is given on the rail resistivity or the DC resistance of the track. However the measurements show a distinct trend at low frequencies towards a DC value of 0.37Ω and this is given if \( \rho = 2.8 \times 10^{-6} \) Ωm. However this is unrealistically high being about 10 times the usual rail resistivity (see Appendix 1). The authors do not say how the rails were shorted but if with a railway wagon the contact resistance between the wheels and the rails could have been significant. If it is assumed that this is 0.18 Ω and this is subtracted from the measurements then the results are consistent with a resistivity of 0.25 \( 10^{-6} \) Ωm (the average resistivity see Appendix 1). This then gives the following:

![Figure 6.2.3.1 Comparison Theory and Measurements Resistance](image)

The magnetic parameters were optimized for best agreement with the measurements and this gave \( \mu_1 = 160 \), \( \mu_\infty = 1 \), \( f_m = 700\text{Hz} \) and \( n=2 \) (Equation 2.6). With these assumptions it is seen that the correlation is very good over the whole frequency range. Also shown is the resistance contributed by the magnetic loss (Equation 2.3), and as with the Haryliuk measurements this is nearly half the total resistance at 1kHz.

The effect of the track-bed impedance on the short-circuited measurements is shown below:

![Figure 6.2.3.2 Short Circuit measurements with and without Track Impedance](image)

The graph shown in red is the theoretical model of the measured values including the track-bed impedance (Equation 5.1.3). In blue is the theoretical resistance of the rails only (Equation 3.3.2). It is seen that the major effect of the track-bed is at frequencies above 250 Hz in this case. Given that the track-bed impedance is in parallel with the shorted rails it might have been expected that it would reduce the resistance but that is not so because of the large rail reactance at high frequencies.
Using the same resistivity and permeability as for the resistance calculations, the comparison between the measurements of inductance and Equation 5.1.3 (the imaginary part) is:

![Inductance of 540m of Track](image)

*Figure 6.2.3.3 Comparison theory and measurements: Inductance*

The track-bed magnetic permeability was set to $\mu_{RT} = 1.15$ (Equation 4.4.1), the same as for the Havryliuk data in Section 6.1. Correlation with measurements is very good over the whole frequency range with the exception of the lowest frequency measurement. Given that this point lies outside the obvious trend of the other measurements it is likely to be an experimental error, particularly since the reactance was only 0.44Ω at this frequency and therefore difficult to measure.

At 50 Hz the inductance is 2.9 µH per metre of track.

A plot of the reactance and the resistance is shown below for the combined rail and track-bed impedances. This has been scaled for 1 km of track:

![Resistance & Reactance for 1Km of Track](image)

*Figure 6.2.3.3 Modeled Reactance and Resistance*

**6.3. Kolar et al**

Kolar et al (ref 15) measured the input impedance of a short UIC 60 rail segment only 1m long. This appears to be that of a single rail of this length rather than a track of this length. They measured the DC resistance as 30 µΩ giving the resistivity as $0.233 \times 10^{-6}$ Ωm, for the rail cross-sectional area of 76.7 cm². No information on rail permeability is given.

The authors were interested in the resistance of the rail to the harmonics of 50 Hz traction currents and they give a graph of the measured impedance and the angle in degrees, for frequencies between 50Hz and 650
Hz, for a drive current at these frequencies of 100A. From this graph the resistance and inductive reactance have been calculated here using the following equations:

\[
R = |Z| \cos \theta \\
X = |Z| \sin \theta
\]

6.3.1

6.3.2

The measured resistance and modelled resistance are given below:

![Resistance of 1 metre Steel Rail](image)

*Figure 6.3.1 Comparison Theory and Measurement*

For the model the permeability was optimised for best match with their measurements, and this was achieved with \( \mu_l = 80, \mu_\infty =1, f_m =400 \text{ Hz} \) and \( n=1 \). It is seen that the correlation with the measurements is very good. Notice the high contribution from magnetic loss.

The measured inductive reactance is very much lower than any reasonable expectations, and the test method is suspect because the authors have not included the mutual inductance between the potential circuit and the current circuit and this must be included (Grover ref 17, p44).
7. COMPARISON WITH IVANEK’S ANALYSIS

7.1. Introduction
The author has given above a different approach to the analysis of track impedance than is generally the case, and here it is compared with the analysis given by Ivanek et al for their measurements.

7.2. Rail Resistance
The track resistance (ie two rails) is shown below, for Ivanek and the author:

![Figure 7.2.1 Track Resistance According to Ivanek for 1km](image1)

![Figure 7.2.2 Track Resistance from author for 1km](image2)

At high frequencies the Ivanek curve shows the resistance decreasing. This is most unlikely because the resistance is determined by skin effect and so we can expect the resistance to continue rising, enhanced by the rising magnetic loss as shown by the author’s data. The Ivanek data is also suspect at low frequencies: at 1Hz they give a resistance of around 650 mΩ whereas the most likely value is much closer to the DC resistance of around 35 mΩ and the author’s curve shows 90 mΩ at 1 Hz.
7.3. Track Inductance
The track inductance is shown below for Ivanek and the author using the same scales for the x and y axes.

![Figure 7.3.1 Track Inductance According to Ivanek for 1km](image)

![Figure 7.3.2 Track Inductance from author for 1km](image)

The agreement here is much better, but with Ivanek’s curve consistently greater than the author’s. In judging which is the most accurate it is significant that the analysis here shows that at high frequencies the inductance should be convergent on that of the external inductance of the track, which for the size of rails and track width here has a value of 1.5 mH. The author’s data does indeed converge on this value but the Ivanek data seems unlikely to do so.

7.4. Track-bed Capacitance
Ivanek gives the following curve for the rail to rail capacitance.

![Figure 7.4.1 Track-bed Capacitance According to Ivanek for 1km](image)
This shows the capacitance reducing from 150 $\mu$F at low frequency to close to zero at high frequencies, a reduction of perhaps 100:1. If this were true it would be very difficult to explain the underlying physical process. In contrast the author gives the following equivalent circuit based on physical components in a concrete sleeper with a porous composition, and where the values are constant, independent of frequency.

![Equivalent Circuit Diagram]

The component values are $R_e = 6.5 \, \Omega$, $R = 5 \Omega$, $C = 3600 \, \mu$F for a 1 km track.

8. ERRORS IN THE ANALYSIS AND MEASUREMENTS

8.1. Limitations of Practical Measurements
Unfortunately there are not many published measurements, and those that are available are often missing important information such as the profile of the rails, their DC resistance and magnetic permeability. Also often missing is the state of the track-bed including the type of sleeper (wood or concrete), the type of gravel bed, its thickness, age and degree of contamination. The weather has a major impact on the resistance of the track bed, including recent weather history and this is often missing. Indeed in order to obtain more accurate measurements of the rails it would be preferable if track measurements were taken in the summer during a dry period since this would minimise the effect of the track-bed. Details of the experimental technique are often very scant and for instance omit to say how the connections to the track were made and how the rails were shorted since this could add resistance. Often little information is given on the measurement equipment or an assessment of the overall accuracy of the measurements. Because of these experimental limitations it has not been possible to assess the accuracy of the theory against the measurements, and it is left to the reader to make a judgement from the information in Section 6. However the theoretical analysis inevitably makes some assumptions, and the effect these have are considered below.

8.2. Open-circuit and Short-circuit analysis
Most measurements are over a short section of track which has been isolated electrically from the rest of the track. At one end of the section two measurements are then taken of the input impedance firstly with the rails shorted together at the far end and secondly with short removed. If the track-bed impedance is high compared with the rail impedance these two measurements then give respectively the rail impedance and the rail to rail track impedance and these are the values required by Equation 5.1.3. However in practice the open and short-circuited measurements are compromised because the track-bed impedance is not sufficiently greater than the rail impedance. The errors are generally acceptable at low frequencies (say below 1 KHz) where the rail impedance is low and high track-bed impedance highest, but these errors increase with frequency because the rail impedance rises and the track impedance falls. The errors are analysed in Appendix 10, and are given below for various ratios of rail impedance to track impedance. The graph below shows the error due to the track-bed impedance in assuming that the short-circuit measurement gives the rail impedance (see Appendix 10).
So for instance taking the Ivanek measurements the short-circuit impedance of the 540 m track at 1 kHz was about 5 Ω (mainly reactance) and the track-bed was 12 Ω and so the error in the rail impedance in using Equation 5.1.3 to allow for the track-bed impedance will be -12%.

The graph below shows the error in the measurement of track-bed impedance (ie the open-circuit case) due to the rail impedance (expressed as a ratio to the track-bed impedance):

So for instance with the Ivanek measurements the error in the measurement of the track-bed impedance due to the rail impedance will be +10% at 1kHz.

In principle the errors could be minimised by modifying Equation 5.1.3, but this would come at the cost of greatly increased complexity. This might not be worthwhile because in practice there are large uncertainties in the magnetic permeability of the rails and the state of the track-bed including gravel degradation (ref 25). So the main value in the theoretical model in the absence of this information is to show the relationships between the various parameters, rather than to give an accurate prediction of a particular track.
9. SUMMARY

9.1. Lumped-element Analysis
The normal method of analysing rail resistance and inductance is to use transmission-line software, but it is shown that this does not model rail-track very well. While it would be possible to improve these transmission-line programs it is shown here that this not necessary and lumped element analysis is sufficient, and can easily incorporate complicated models of the rail and track-bed.

9.2. Rail Resistance
It has been shown that the resistance of the rails is dominated by skin-effect and this is dependent upon the resistivity and the magnetic permeability. Also at the higher frequencies the magnetic loss in the rails can be as significant as the resistive loss. So given these dependencies it is essential that the magnetic characteristics of the rail are known if predictions are to be made of the rail resistance.

9.3. Track Inductance
The track inductance is shown to be comprised of three factors:
   a) the external inductance of the rails,
   b) a small increase in this inductance due to the magnetic permeability of the track-bed, and
   c) the internal inductance of the rails.
The external inductance is due to the magnetic field around the rails and the field between them, across the track-bed. This inductance is independent of frequency and dependent only on the geometry of the track, the rail cross-section and the rail spacing. It is increased by about 15% due to the permeability of the track-bed.
The internal inductance is frequency dependent because the penetration of magnetic flux into the rail reduces as the frequency increases. At low frequencies there is large penetration of flux into the rail and the internal inductance can then exceed the external inductance.

9.4. Track-bed Resistance and Capacitance
The track-bed resistance and capacitance are normally modelled as a resistance and capacitance in parallel, but this leads to claims that the capacitance varies by perhaps 100:1 with frequency. It is shown here that if the track-bed is modelled by a series/parallel network the capacitance is constant with frequency.
Appendix 1 : ELECTRICAL PARAMETERS OF RAILS

A1.1. Resistivity and Magnetic Permeability

The resistivity and magnetic permeability of rails is not readily available. A search of the web gives:

<table>
<thead>
<tr>
<th>Author</th>
<th>Rail Type</th>
<th>Steel</th>
<th>Resistivity (Ωm)</th>
<th>Permeability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Havryliuk (Czech, ref1)</td>
<td>R65</td>
<td>Steel</td>
<td>0.21 10⁻⁶</td>
<td>100</td>
</tr>
<tr>
<td>Kolar (Czech republic, ref 15)</td>
<td>UIC 60</td>
<td>R65</td>
<td>0.25 10⁻⁶</td>
<td>85</td>
</tr>
<tr>
<td></td>
<td>T</td>
<td>Steel</td>
<td>0.25 10⁻⁶</td>
<td></td>
</tr>
<tr>
<td></td>
<td>S49</td>
<td></td>
<td>0.25 10⁻⁶</td>
<td></td>
</tr>
<tr>
<td>Kiraga &amp; Szychta (Poland)</td>
<td>UIC 60</td>
<td>R260</td>
<td>0.25 10⁻⁶</td>
<td>95</td>
</tr>
<tr>
<td></td>
<td>&amp; R350 HT</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Teng et al (China)</td>
<td></td>
<td></td>
<td>1.1 10⁻⁶</td>
<td></td>
</tr>
<tr>
<td>Mierczak (UK)</td>
<td>R260</td>
<td>Steel</td>
<td>0.22 10⁻⁶</td>
<td>90</td>
</tr>
<tr>
<td>Mehboob</td>
<td>R260</td>
<td>Steel</td>
<td>180 (μᵣ)</td>
<td></td>
</tr>
<tr>
<td>Hill &amp; McKay (UK)</td>
<td></td>
<td></td>
<td>0.23 10⁻⁶</td>
<td>380 (running rail)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.23 10⁻⁶</td>
<td>1590 (power rail)</td>
</tr>
<tr>
<td>Szychta (Poland, ref 3)</td>
<td>60 E1 (UIC 60)</td>
<td>R260 &amp; R350 HT</td>
<td>40(μᵣ), 180 max</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.26 10⁻⁶</td>
<td>(Edge of web)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.285 10⁻⁶</td>
<td>(Centre of web)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.275 10⁻⁶</td>
<td>(Foot)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.257 10⁻⁶</td>
<td>(Taper of foot)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.27 10⁻⁶</td>
<td>(Taper of web)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.26 10⁻⁶</td>
<td>(Rail head)</td>
</tr>
<tr>
<td>RailCorp Spec SPG 0709</td>
<td></td>
<td></td>
<td>0.25 to 0.28 10⁻⁶</td>
<td>depending on rail</td>
</tr>
<tr>
<td>Hideki et al</td>
<td>50kg N rail</td>
<td></td>
<td>0.203 10⁻⁶</td>
<td>70 (μᵣ)</td>
</tr>
</tbody>
</table>

A1.2. Types of Steel for Rails

The steel designations R260 and R350HT are European EN standards, where R is for rail, and the numbers refer to the Brinell hardness. HT is heat treated.

Another commonly used steel is Grade 880 and this is to an IRS specification (Indian Railway Standard).

A1.3. Rail Resistance

From Section A 1.1, the likely range of rail resistivity is in the range 0.25 10⁻⁶ ± 25%. The rail resistance will therefore be for the following rail types:
Some support for the above is given by the Railtrack Standard reference 22 page 9, which states for running rails: “The resistance per unit length of rail depends on the type of rail and the degree of wear and corrosion. For a single rail, the range from new 60kg/m rail to fully worn 47 kg/m rail is 29 to 40 mΩ/km. Also Bongiorno & Mariscotti (ref 20) give a measured value of 43 mΩ/km, but they do not give the rail type.

The track resistance will be twice the above values.

The resistivity values above are assumed to be at 25°C. The temperature coefficient for steel is 0.003 / degree C, and so a ± 20°C change in ambient will give a ±6% change. However the temperature of the rails can be 20°C above ambient (reference 19), adding another 6% to the resistance.

### A1.4. Rail Permeability

Reference 3 gives the following curve of permeability versus frequency, measured on samples taken from a 60E1 rail.
So this data suggests $\mu' \approx 40$, $\mu'_{\infty} \approx 25$ and $f_m \approx 2000$ Hz. However, their Figure 13 shows that the permeability is highly dependent on the magnetization $H$:

![Figure A1.4.2 B/H and permeability/ H](image)

So the permeability can be up to 150 at larger values of $H$, and as small as 20 (extrapolating the curve down to small values of $H$).

However these measurements were made by inserting a section of rail into a coil and measuring the increase in inductance. This could give erroneous results because it is known that the permeability of rolled steel is dependent upon the direction of the field compared with the rolling (ref 12). The above measurements were with the flux down the length of the rail whereas the flux due to the current is effectively normal to this.

Mariscotti (ref 22, p1073) says ‘In general, magnetic permeability (at 50Hz) may be assigned values between 25-50 at low current and between 50-100 at high current’.

### A1.5. Rail Inductance

Inductance cannot be calculated for a single rail since most of the magnetic flux exists between the two rails. However having calculated this we can assign a ‘partial inductance’ of half this value to each rail.

Reference 22 (page 9) says ‘The inductance of the ‘out and back’ loop is typically 2.2 to 2.5 mH/km (for DC). However it has been shown here that the inductance is highly frequency dependent and that at low frequencies the internal inductance of the rail can be dominant and so the inductance is highly dependent upon the magnetic permeability, and for instance Figure 6.1.2 shows a range from 4.2 to 1.5 mH/km.

### Appendix 2    RECTANGLE WITH SAME AREA AND PERIPHERY AS RAIL

If the rectangle has a width $w$, and thickness $t$, and the rail has a perimeter $p$ then

$$w + t = \frac{p}{2}$$  \hspace{1cm} A2.1

If the rail has a cross-sectional area $A$, then:  

25
w = A/t \hspace{1cm} \text{A2.2}

Substituting 2.2 into 2.1:

\[ A + t^2 = \frac{p}{2} \] \hspace{1cm} \text{A2.3}

So

\[ t^2 - \frac{p}{2} + A = 0 \] \hspace{1cm} \text{A2.4}

Solving for this quadratic equation gives:

\[ t = \frac{\frac{p}{2} - \sqrt{\left(\frac{p}{2}\right)^2 - 4A}}{2} \] \hspace{1cm} \text{A2.5}

From Equation A2.1

\[ w = \frac{p}{2} - t \] \hspace{1cm} \text{A2.6}

\textbf{Appendix 3} \hspace{1cm} \textbf{TRACK INDUCTANCE}

\textbf{A3.1. Model Track}

Most equations for inductance are for conductors with a circular cross-section, and so to use these for rails an equivalent diameter can be chosen. Often authors have used a diameter which gives the same cross-sectional area as that of the rail, and to test this idea measurements were made of the inductance of a short section of model rail. The track used is called Code 75, and is an approximate scale model of UIC60 track:

![Model Railway track](image)

\textit{Figure A 3.1 Model Railway track}

The following dimensions were measured:
Distance between the \textit{centre} of the rails of 17.4 mm
Rail height : 1.93 mm
Head width : 0.79 mm
Foot width : 1.76 mm
Length of track : 913 mm

This track was shorted out at one end with a short copper wire 25 mm long soldered between the two rails. It was connected at the other end to a VNA via two copper wires each of length 22mm also soldered to the
The inductance was measured over a frequency range of 2 MHz to 8 MHz, chosen to ensure that the impedance was not below 10 Ω, since the measurements are not so accurate below this level, and not too close to the self-resonant frequency measured as 60 MHz (Appendix 7). The measured inductance was corrected for the connection leads, the SRF and the inductance calibration error of 1% and the average over this range was 1.18 µH.

Grover (ref 17 p 39) gives the inductance of this return circuit with circular conductors as:

\[
L = \mu_r \frac{2\ell}{(2\pi)} \left[ \ln \frac{2D}{d} + \mu_r \frac{4-d}{\ell_T} \right]
\]

where
- \ell is the length of each of the two conductors
- D is the centre to centre spacing of the conductors
- d is the diameter of the conductor
- \mu_r is the relative permeability of the conductor

The model rails are probably made from nickel silver so that relative permeability is unity (\mu_r = 1) confirmed by the fact that they were not attracted to a powerful magnet.

If the diameter d is chosen to give the same cross-sectional area as the rail, A, then d = (4A/\pi)\sqrt{\frac{5}{8}}. For this we need to know the area of the rail but this difficult to determine on the small model rail but that of the full size rail is known and so that of the model rail will be smaller by the cube of the scaling factor. The model rail is nominally 3.5 mm to 1 foot ie 1:87, and the area of the full size UIC 60 rail is 76.7 cm² and so that of the model is 76.70 \times 10^{-4}/(87)^3 10^9 = 11.65 mm², and the diameter which gives the same area is 3.85 mm. Using this diameter Equation A3.1 gives an inductance of 0.64 µH, compared with that measured of 1.1 µH, an error of over 40%.

So using the equivalent diameter gives a very poor estimate of the rail inductance.

The alternative equivalent is a rectangular rail with the same cross-sectional area and the same periphery. For a return circuit of two rectangular parallel bars of length ℓ, width w, thickness t and centre spacing D the inductance is given by Terman as (ref 5, p52):

\[
L = \mu_r \frac{2\ell}{(2\pi)} \left[ \ln \frac{D}{w+t} + 1.5 - D/\ell + 0.2235 (w+t)/\ell \right]
\]

The above equation is for a conductor with a relative permeability of unity, so that the contribution to inductance from the field inside the wire is included in the factor 1.5. For a circular conductor this contribution is \frac{1}{4} ie the factor 1.5 is equal to 1.25 + 1/4. For a rectangular conductor the factor is probably closer to 1.4 (ie 1.35+1/6.5) at DC (see Section 4.3) and this is further reduced in the measurements because of skin effect (skin depth was 0.12 mm at 5 MHz).

For the UIC 60 rail A = 76.7 cm² and the periphery is 683 mm, so from Equation 3.1 w = 317 mm and t = 24 mm, which when scaled by 87 gives the model width w = 3.65 mm and t = 0.28 mm.

Using these dimensions in the above equation gives an inductance which is 5.5% lower than the measured value. This can be corrected by increasing the term 1.5 to 1.7.

So a the equation which best describes the inductance of the model rail is:

\[
L_E = \mu_r \frac{\ell_T}{(2\pi)} \left[ \ln \frac{D}{w+t} + 1.7 \right]
\]

[When the rail dimensions and spacing are small compared with the length as here, the last two factors in the Equation A3.1.2 increased the inductance by only 0.5% and so can be neglected].

This equation includes the internal inductance of the model rail but this inductance would have been small because the rail material was non-magnetic and the measurement frequency was high enough for the current penetration to be small compared with the rail dimensions (skin effect). So it is assumed that the above equation gives a close approximation to the external inductance L_E of rails.

A3.2. Track-bed Magnetic Permeability

With a full-size track the inductance will be greater than the model because the track-bed has a magnetic permeability slightly greater than unity. This increased permeability is due to metal between the rails not
present in the model track above, including fastening plates and metal tie rods in concrete sleepers. Also the magnetic permeability of gravel can be slightly above unity (ref 28). Good agreement with measurements on real track is given if $\mu_{RT} \approx 1.15$ (see Section 6).

Equation A3.3 then becomes:

$$L_E = \mu_{RT} \mu_o \ell_T/(2\pi) \left[ \ln D/(w+t) + 1.7 \right] \text{ Henrys}$$

where

- $D$ is the centre to centre distance between rails
- $w$ and $t$ are given by Equation 3.1.1
- $\ell_T$ is the length of the track
- $\mu_o \approx 4\pi \times 10^{-7} \text{ H/m}$
- $\mu_{RT}$ is the relative permeability of the track-bed $\approx 1.15$

**Appendix 4  INTERNAL PERMEABILITY FACTOR $k_P$**

**A4.1. Introduction**

The internal inductance of a rectangular conductor is assumed to be given by Equation 4.3.2, repeated below

$$L_I \approx \mu_o 2\ell / (2\pi) \left( \mu_R / k_P \right) \left( 1 - e^{-x} \right) / K_c \text{ Henrys}$$

In this equation the effective permeability at DC is $(\mu_R / k_P)$. The value of $k_P$ is unknown, although an example given by Antonini et al (ref 10) corresponds to $k_P = 6.3$ for a rectangular conductor with $w/t = 10.7$. Also for a circular conductor it is known that $k_P = 4$. [NB for a circular conductor $K_c = 1$ and the value of $x$ in $(1 - e^{-x})$ is different to that of the rectangular conductor- see ref 6].

To add to the data on the value of $k_P$ the internal inductance of steel tape was measured. This tape had a length of 18.3 meters, width of 9.7 mm and thickness 0.4 mm, so $w/t = 24.3$. The resistivity was determined as $2 \times 10^{-7} \Omega \text{ m}$, calculated from the measured resistance of 0.95 $\Omega$.

The steel strip was bought on e-bay and sold as ‘steel boning’ for use in corsets. This had the advantage of being covered in a plastic film, ($\approx 0.3 \text{ mm thick}$) which made it much safer to handle than bare strip.

**A4.2. Steel Strip with w/t = 24.3**

The external inductance of the strip can be large and mask the internal inductance and so the strip was folded back on itself to minimize this external inductance, giving a folded length of 9.15 metres.

The strips were held together with adhesive tape spaced every 300 mm along the length. One end of the parallel strips is shown below:

*Figure A 4.2.1 One end of folded strip with connections*
The plastic covering provided convenient insulation between the two halves. The inductance was then measured from 10 kHz to 2 MHz, the jig inductance subtracted from this and the result corrected for the SRF measured as 3.8 MHz (see Appendix 7). The external inductance is not totally cancelled by the folding, but the remnant value is difficult to calculate with accuracy and so has been measured. This was done by measuring the inductance of the folded strip at a high frequency where the permeability of the steel had dropped to a low value and the skin depth was small. These measurements were then extrapolated to zero skin depth when the internal inductance would be zero and from this the external inductance was found to be 1.18 µH (details are given later). This value was subtracted from all the measurements to give the measured internal inductance, shown below in brown:

![Figure A 4.2.2 Measurements of internal inductance and that calculated](image)

Also shown is Equation 4.3.2 (in red) programmed with the permeability as modeled in Appendix A5.2. The value of $k_p$ was adjusted for the best agreement with the measurements and this achieved with $k_p = 12$. The poor correlation at low frequencies is partly due to measurement error (the reactance was only 0.6 Ω at 10 kHz) and partly because the Q of the strips dropped to only 0.5 at 10 kHz.

[To find the residual inductance of the folded conductor, its inductance was measured at high frequencies where the internal inductance will have reduced to a low value because the low current penetration (skin effect) will have reduced the effect of the steel permeability. In addition the permeability will also have dropped to a low level. To measure at a higher frequency the length of the steel pair had to be reduced to 3.025 meters to minimize the correction needed for SRF. The measured values are given below:

![Figure A 4.2.3 Inductance at high frequencies](image)

In the above, the jig inductance (0.2 µH) has been subtracted from the measured values and the resultant value corrected for SRF (measured as 24.7 MHz). As the frequency is raised the inductance reduces...
because the skin depth is reducing (the permeability is constant at 12, see A5.2). To determine the value of inductance at zero skin depth the inductance was plotted against skin depth as below:

![Remnant Inductance](image)

*Figure A 4.2.4 Extrapolation to zero skin depth*

This data is extrapolated to zero skin depth where the internal inductance will be zero, to give 0.39 µH. When scaled to the 9.15 meter length gives $0.392 \times 9.15/3.025 = 1.18$ µH.

### A4.3. Steel Strip with w/t = 12.1

In the above experiment the w/t ratio was 24.3. To produce a conductor with a smaller w/t ratio the strip was double up to give twice the thickness, and w/t = 12.1. For this the insulation was stripped off and two bare strips placed back to back and touching so that the total conductor thickness was $2 \times 0.4 = 0.8$ mm. Two such ‘double’ conductors were placed back to back with plastic adhesive tape between them to provide insulation. The total length of this structure was 3.18 metres, so that the total conductor length was 6.36 meters.

The inductance was then measured from 20 kHz to 5 MHz, and the jig inductance subtracted from this and the result corrected for the SRF measured as 8.8 MHz (see Appendix 7). The external inductance is not totally cancelled by the folding, and this had been measured in A4.2, and was scaled for the shorter length here to give 0.41 µH. This value was subtracted from all the measurements to give the measured internal inductance, shown below in brown:

![Internal Inductance](image)

*Figure A 4.3.1 Measurements of internal inductance and that calculated*
Also shown is Equation 4.3.2 (in red) programmed with the permeability as modeled in Appendix A5.2. Best agreement with measurements was with \( k_p = 8 \), and this is the value used in the plot above.

A4.4. Factor \( k_p \) summary

The above measurements are shown below in red:

![Internal Permeability Factor \( k_p \)](image)

Figure A 4.4.1 Data on Permeability Factor \( k_p \)

Also shown is Antonini’s data in blue, and the known value for the circular conductor of 4, plotted at \( w/t = 0 \). The straight line curve is given by:

\[
 k_p \approx 4 + 0.3(w/t) \tag{A4.4.1}
\]

For the UIC 60 and R65 rails \( w/t \) is equal to 11.9 and so \( k_p = 7.9 \), and this the value used in the comparisons with published measurements Section 6.

Appendix 5 MEASUREMENT OF STRIP PERMEABILITY

A5.1. Introduction
The above section determined the internal permeability factor \( k_p \) and this required the permeability of the steel strip to be measured. These measurements are described here.

It is known that when steel is rolled during manufacture the permeability can be different in the three dimensions with often a much higher permeability in the rolled direction (Hihat et al ref 12).

When current flows down the strip the magnetic field is normal to the length of the strip (which is most likely the rolled direction) and so measurements were made with flux in that direction. Eddy currents induced in the metal reduce the flux passing through and thus reduce the apparent permeability. This is considered in Appendix 11, and the measurements below have been corrected for this.

A5.2. Measurement of Permeability in direction normal to length of strip

Flux was generated in a gap in a closed ferrite core as shown below.
The core was designated UY1660, 60 mm square by 16 mm diameter, material MnZn with a permeability of around 2000 and made by the Kefeng company China. The strips were placed on edge in the gap, with 35 strips in each gap, and filed as one to ensure they were the same width. The strips were not insulated from one another other than by any natural oxidation, and they were held together by plastic adhesive tape stretched tightly over them to maintain a pressure. The total width of the strips was thus chosen to be slightly greater than the diameter of the core. There was a small gap between each strip and the packing was determined to be 0.85. Downwards pressure on the core was maintained with an elastic band around the two cores (not shown). The inductance of the winding was then measured from 5kHz to 1 MHz with the winding shown. At higher frequencies the effect of self-resonance degraded the accuracy and so a winding with fewer turns was used to measure at 2 MHz and 5 MHz. To check that the ferrite core maintained its reluctance over the frequency range the inductance with a 0.2mm air gap was measured over the frequency with no significant change.

To calibrate the equipment the reluctance of the steel strips was compared with that of an air gap. For this the strips were removed and various thicknesses of paper were substituted until one was found which gave the same inductance as with the strips present. The permeability of the steel was then equal to the ratio of the steel dimension (10.19 mm) and that of the air gap, found to be 0.12 mm, giving the steel permeability as 83.8 at 5 kHz (the assumption here was that the reluctance of the ferrite core was very much lower than that of the air gap, so that the total reluctance was determined by the length of the air gap alone).

The inductance at 5 kHz was 401 µH and so an inductance of L_M measured at any other frequency would correspond to a permeability of $\mu' = 83.8 \frac{L_M}{401}$. These values were corrected for packing density, eddy-current reduction and SRF (Appendix 7).

The measured permeability is shown below in blue:
Also shown in red is Equation 2.6 with the parameters optimized for best match with the measurements, and this was achieved with \( \mu_1' = 93, \mu_\infty' = 25, f_m = 41 \text{ KHz}, \) and \( n = 1.15. \)

**Appendix 6 MEASUREMENT APPARATUS AND ACCURACY**

All measurements were made with an Array Solutions UHF Vector Network Analyser. Calibration of this analyser required an open circuit, a short circuit and known resistive load, and these are shown below.

To ensure that the calibration resistance had minimal stray reactance a thick-film resistor was used (above), and this had the added advantage that it could be located in the same plane as the short circuit. Its value was \( 47 \Omega \pm 1\%. \) SMA connectors were used because they are small and therefore have a small stray capacitance, and so any error in calibrating this out would also be small. To test the accuracy of the inductance measurements copper wire of diameter 0.4 mm was formed into a single turn coil with a perimeter of 467 mm (including the 3mm gap of the SMA connector), so its diameter was 148.7 mm, as shown below:

---

**Figure A 6.1 Calibration loads**

---

**Figure A 6.2 Calibration loads**

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A thin conductor is used to avoid the uncertainty of the diameter of current flow inherent in a thick conductor. The theoretical inductance of this loop is 0.560 µH, found from:

\[ L_{\text{LOOP}} = N^2 R \mu_0 \mu_R (\ln 8 R/a - 2) \text{ Henrys} \]  \hspace{1cm} \text{A6.1}

where

- \( N \) is the number of turns
- \( R \) is the radius of the loop in metres
- \( a \) is the radius of the wire in metres
- \( \mu_R = 1 \)

This loop was measured on the VNA with the following results:

\[
\begin{array}{cccc}
\text{Frequency MHz} & \text{Measured L µH} \\
0.1 & 0.1 \\
1.0 & 1.0 \\
10.0 & 10.0 \\
100.0 & 100.0 \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{Loop Inductance} \\
\text{Measured L µH} \\
\text{Calculated L} \\
\text{Measured L Cal 1} \\
\end{array}
\]

Above 1 MHz the error is less than 3 %. At the lower frequencies the error increases, largely because the reactance was very low (less than 1Ω below 0.3 MHz).

Appendix 7: CORRECTIONS FOR SELF-RESONANCE

The folded conductor constituted a two-wire transmission-line and this resonates when its folded length is equal to \( n\lambda/4 \), where \( \lambda \) is the wavelength. Thus the first resonant frequency is when

\[ f_R = \frac{300}{4 \ell} \]  \hspace{1cm} \text{A 7.1}

where \( \ell \) is the length of the folded line.

As this frequency is approached the measured resistance and the inductance increase above their low frequency value and Welsby (ref 16, p 37) has shown that this is given by:

\[ L = L_M \left[ 1 - \left( \frac{f}{f_R} \right)^2 \right] \]  \hspace{1cm} \text{A 7.2}

\[ R = R_M \left[ 1 - \left( \frac{f}{f_R} \right)^2 \right]^2 \]  \hspace{1cm} \text{A 7.3}
L_M and R_M are the measured values, and f_R is the self-resonant frequency. Welsby developed these equations for the self-resonance in coils and they are less accurate for a transmission-line and become increasingly inaccurate as the resonant frequency is approached. In practice it is better to measure the self-resonant frequency in the measurement jig rather than calculate it from Equation A7.1 because a very small stray capacitance in the jig will reduce the SRF considerably. The SRF is defined here as the frequency at which the impedance goes through zero phase angle.

Appendix 8  TRACK-BED MEASUREMENTS

A8.1.  Introduction
Ideally a physical description of the tack-bed would be sufficient to determine the track bed parameters for the theoretical model, Figure 5.2.1. However this is very difficult because the possible values are very large as is the uncertainty. This problem is illustrated by the following information on resistance and capacitance obtained from scanning the internet.

Most measurements have been made at DC and so are of resistance only. The only AC measurements known to the author are by Ivanek (ref 14).

A8.2.  Sleeper Resistance
The European Standard DIN EN 13146-5 specifies a minimum resistance of 5kΩ for each sleeper. It specifies a laboratory test procedure for determining the electrical resistance in wet conditions, between the running rails provided by a fastening system fitted to a steel or concrete sleeper, bearer or element of slab track. It is also applicable to embedded rail. Assuming 1500 sleepers per km the combined resistance will be a minimum of 3.3 Ω·km.

Traditionally creosoted wood has been used for sleepers and this has a resistivity between 2.5 × 10^3 Ωm (when wet) and 10^7 with 20% moisture, so this would give a rail to rail resistance of between 80Ω and 32 kΩ over 1 km. However the surrounding ballast can have a lower resistivity and this will lower the rail to rail resistance.

A8.3.  Measured Ballast Resistivity
Neupane (ref 25) has measured the reduction in ballast resistivity as it becomes dirty and finds for clean ballast greater than 4400 Ωm (value limited by his measuring equipment), reducing to an average of 160 Ωm (p161) for dirty ballast. Thus the degradation is by a ratio of greater than 28. However to translate this into rail to rail impedance we must know the impedance between the rail and the ballast via the fixings and sleeper.

A8.4.  Measured Rail to Rail Resistance
Landau (ref 24) reports a survey of the rail to rail resistance for a rail transit system in the eastern U.S. Outside surface-running areas were generally in a range of 7.5 to over 350 Ω·km. This is a ratio of 46, and consistent with Neupane’s findings for the degradation of ballast. Tunnels had the lowest ballast resistance probably due to accumulation of steel dust, poor drainage and seepage. Some of the lowest values were in river tunnels beneath salty estuary water. Tunnel values were mostly well below 350 Ω·km with the majority of those below 7.5 Ω·km; many were below 1.5 Ω·km; and some were as low as 0.06 Ω·km.

Unfortunately he does not say whether the sleepers were wood or concrete.

A8.5.  Effect of Insulating the Rails
Landau (ref 24) reports that replacing the existing tie plates with insulated tie plates improved the resistivity from a range of 0.3 to 3 Ω·km to a range of 80 to 200 Ω·km. This clearly shows that conduction is mainly via the sleepers and any direct contact between the rails and the ballast is very small in comparison.
A8.6. Rail to Rail Capacitance

The reported values of rail to rail capacitance are very high with Hill & McKay giving ‘practical values range from 0.1 to 100 µF/km’. An even higher value of 3600 µF/km is deduced for the Ivanek data (A8.2). These are far higher than the rail to rail capacitance via the air which is around 0.01 µF for two rails 1km long and in isolation. However some of the electric field will be in the gravel track-bed and this will increase the capacitance by the average dielectric constant of air and gravel:

\[ \varepsilon_{\text{EFF}} \approx \frac{(\varepsilon_R+1)}{2} \]

where \( \varepsilon_R \) is that of the gravel.

The dielectric constant of gravel is from 3 to 20 depending upon wetness (water has an \( \varepsilon_R \) of 80) and so the effective dielectric constant will be from 2 to 10.5 from the above equation. So the rail to rail capacitance through the gravel/air will be from 0.03 to 0.2 µF.km. This is still much smaller than the reported values and so we need to consider the capacitance across the sleepers. It is known that high capacitance values can arise from the porous nature of concrete (Song ref 26). The diagram below shows the microstructure of concrete, with solid material shown in grey and pores shown in light blue.

Figure A 8.6.1 Microstructure of concrete

The solid material has a high resistance. The pores have a much lower resistance because they are filled with conducting liquid and conduction occurs by ion migration and therefore obeys Ohm’s law. There are therefore three paths for current:

a) high resistance paths consisting only of solid material  
b) low resistance paths consisting only of conducting liquid  
c) paths which are mainly conducting liquid blocked by a short piece of solid material. These give the high capacitance values.

Clearly if the concrete is very wet the conducting paths will have a lower resistance. Also the concrete has metal strengthening rods along its length and so these will further reduce the resistance and increase the capacitance.

A8.7. Track-bed Magnetic Permeability

See appendix A3.2

Appendix 9 1/3rd FACTOR FOR TRACK-BED IMPEDANCE

When a section of track is shorted at the far and a signal injected at the open end the voltage across the rails will decrease linearly to zero at the shorted end (see Note 1). This is shown in the diagram below:
If the total resistance *between* the rails is $R$, then over a distance $\delta x$ the resistance will be:

$$R_X = \frac{R \ell}{\delta x}$$  \hspace{1cm} \text{(A9.1)}

where $\ell$ is the length of the track.

If the applied voltage is $V$ then the voltage at distance $x$ is:

$$e = V \left(1 - \frac{x}{\ell}\right)$$  \hspace{1cm} \text{(A9.2)}

thus the power dissipation over $\delta x$ is:

$$P_X = \frac{e^2}{R_X} = \left[\frac{V^2}{R \ell}\right] \left(1 - \frac{x}{\ell}\right)^2 \delta x$$  \hspace{1cm} \text{(A9.3)}

So the total power dissipation is:

$$P_T = \left[\frac{V^2}{R \ell}\right] \int_0^\ell \left(1 - \frac{x}{\ell}\right)^2 \delta x = \frac{V^2}{3R}$$  \hspace{1cm} \text{(A9.4)}

So the effective resistance is 3 times the rail to rail resistance. By a similar analysis it can be shown that the capacitance is reduced by a factor of 3 (ref 21, p232).

(Note 1: This is essentially true if the length of the track is much less than a wavelength, and so this is the same assumption as that in Section 1 for using lumped components.)

**Appendix 10 ANALYSIS OF ERRORS**

A10.1. General
Equation 5.1.3 is valid when the rail impedance is small compared with the track-bed impedance. The errors if this is not true are analysed below.

A10.2. Short Circuited Measurement
To measure the rail impedance a section of track is shorted at the far end the impedance at the open end measured. The track-bed impedance appears in parallel with the rail impedance as shown below:
The track-bed impedance is increased by a factor of 3 as shown in Appendix 9. The input impedance is then:

\[
Z_{IN} = \frac{3 Z_T Z_R}{3Z_T + Z_R}
\]

For the rail impedance \( Z_R = 1 \)

\[
Z_{IN} = \frac{3 Z_T}{3Z_T + 1}
\]

So when the track impedance \( Z_T = 1 \) the input impedance is 0.75, and error of -25% on the rail impedance. For other values:

\[
Z_{IN} = \frac{3 Z_T Z_R}{3Z_T + Z_R}
\]

\[
Z_{IN} = \frac{3 Z_T}{3Z_T + 1}
\]

A10.3. **Open Circuited Measurement**

To measure the track-bed impedance the input impedance of an open-circuited section of track is measured. The rail impedance appears in series as shown below:
Notice that in the above equivalent circuit the track impedance is split into two impedances of 2 $Z_T$ each.

The input impedance is:

$$Z_{IN} = \frac{2Z_T(Z_R+2Z_T)}{(2Z_T+Z_R+2Z_T)} = \frac{(2Z_T Z_R + 4 Z_T^2)}{(Z_R+4Z_T)}$$

If $Z_T = 1$

$$Z_{IN} = (2Z_R+4)/(Z_R+4)$$

So when $Z_R = 1$ the $Z_{in} = 1.2$ or + 20% of the track-bed impedance. For other values of rail resistance the error is:

\[\text{Figure A 10.3.2 Error in Track-bed Impedance}\]

**Appendix 11 REDUCTION OF FLUX DUE TO EDDY CURRENTS**

Eddy currents induced in a conductor generate a flux which opposes the incident flux, so that the resultant flux is reduced. It can be shown that the ratio of the resultant flux to the incident flux is given by (ref 30):

$$\frac{B_r}{B_o} = \frac{1}{1+\frac{3}{\mu'}}$$

\[\text{where } \mu' \text{ is the relative permeability of the conductor}\]

This equation applies when the conductor thickness is greater than 3 skin depths. Notice that surprisingly it is independent of the conductor resistivity.

In applying this to the measurement of permeability it is the magnetic susceptibility which is reduced and so we have:

$$\frac{\mu_M-1}{\mu'-1} = \frac{1}{1+\frac{3}{\mu'-1}}$$

\[\text{A11.2}\]

$$\mu' = (\mu_M-1) \left[1+\frac{3}{\mu'-1}\right]$$

\[\text{where } \mu_M \text{ is the measured permeability of the conductor}\]

Multiplying by $(\mu'-1)$ gives

$$(\mu'-1)^2 = (\mu_M-1) (\mu'-1) + 3 (\mu_M-1)$$

\[\text{A11.3}\]

This is a quadratic equation whose solution is:
\[ \mu' = \left[ (\mu_{M} - 1) + \left( (\mu_{M} - 1)^2 + 12 (\mu_{M} - 1) \right)^{0.5} \right] / 2 + 1 \]

This is plotted below:

![Material Permeability vs Measured Permeability](image)

*Figure A 11.1 Material \( \mu \) versus measured \( \mu \)*
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