THE INDUCTANCE OF A SINGLE LAYER COIL DERIVED FROM CAPACITANCE

The inductance of a coil can be derived from the magnetic reluctance to its flux, and for a single layer coil this reluctance can be derived from simple equations for capacitance. These equations can then be used to solve the more difficult problem of the inductance of a coil when a ferrite rod is inserted.

1. INTRODUCTION

This article shows that an equation for the inductance of a coil can be derived from the reluctance of the magnetic paths inside and outside the coil, using familiar equations for capacitance. This is of more than academic interest, because the techniques developed here can be used to solve the more difficult problem of the inductance of a coil when a ferrite rod is inserted, and this is the topic of a later article (ref 5).
The analysis here uses some basic magnetic theory and this is addressed in the next section.

2. SIMPLE MAGNETIC THEORY

A coil is a component which produces flux when a current flows, and its inductance, L, is the number which relates the amount of flux the coil produces for the given current:

$$L = \Phi/I \qquad \text{henrys} \qquad 2.1$$

Or if the coil has N turns :

$$L = N\Phi/I \qquad \text{henrys} \qquad 2.2$$

Notice that a coil which produces more flux for a given current will have a higher inductance.
The theory in this article uses some basic magnetic theory, particularly the concept of reluctance. Many readers will be unfamiliar with this but magnetic theory is based on Ohm's law, as the following will show.

Figure 1 shows the relationships between the electric circuit and the magnetic circuit.
The *electric* circuit consists of a battery and a series resistance R, and the current I which then flows is given by Ohm's law as :

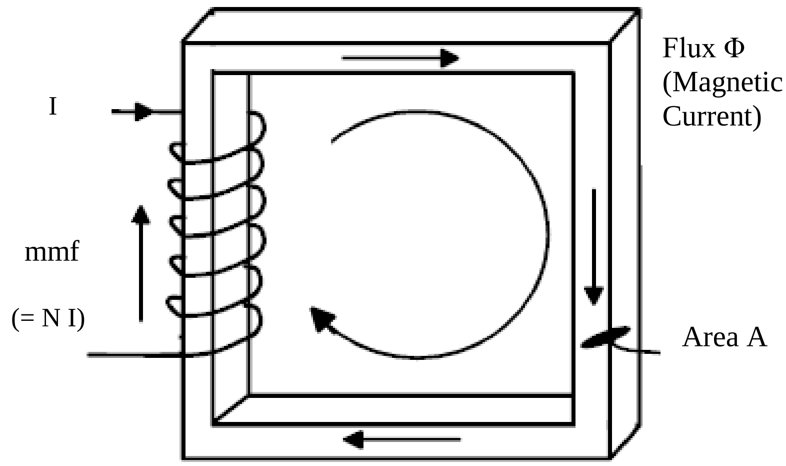
$$I = emf\ /R \qquad 2.3$$

The emf is the electro-motive force of the battery (measured in volts) and it is the pressure which drives current around the circuit against the resistance R.

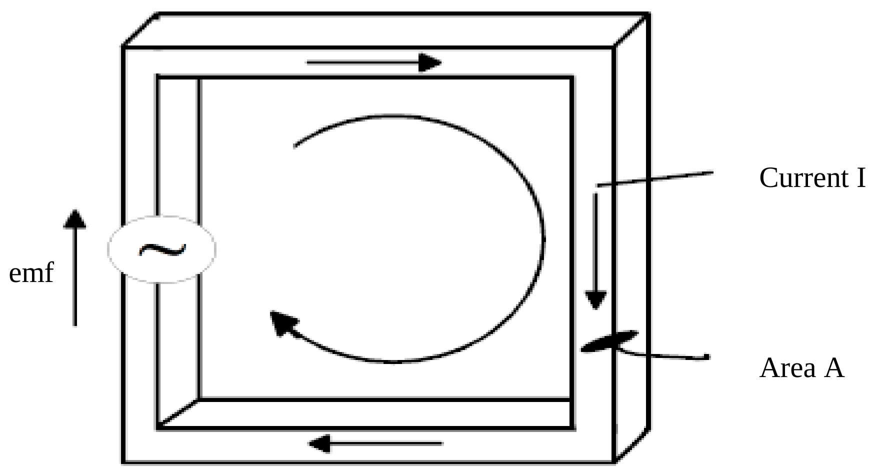
In the magnetic circuit the *magnetic current* is called flux, and the resistance to the flow of flux is called the reluctance $\mathcal{R}$. So the flux Φ is given by :

$$\Phi = mmf/\mathcal{R} \qquad 2.4$$

Notice the similarity between the above two equations.



Flux $\Phi = mmf/\mathcal{R}$
$\mathcal{R} = (1/\mu)\, l/A$



Current $I = emf/R$
$R = \rho\, l/A$

Figure 1   Comparison of Electric and Magnetic Circuits

The mmf is the magneto-motive force which drives flux around the circuit, and it is provided by current flowing through a coil surrounding the magnetic circuit (the magnetic circuit can be air). Its value is :

$$mmf = NI \qquad 2.5$$

where N is the number of turns and I is the current.

Combining equations 2.4 and 2.5 :

$$\Phi = N I / \mathcal{R} \qquad 2.6$$

The inductance is then given by:

$$L = N \Phi / I = N^2 / \mathcal{R} \quad \text{Henrys} \qquad 2.7$$

The inductance is therefore inversely proportional to reluctance and this is very similar to the *resistance* in an electric circuit. The resistance of a conductor is given by :

$$R = \rho\, l / A \qquad 2.8$$

where ρ is the resistivity of the conductor e.g copper
l is the length of the conductor
A is the cross-sectional area of the conductor.

Similarly the reluctance $\mathcal{R}$ of a magnetic circuit is :

$$\mathcal{R} = (1/\mu)\; l / A \qquad 2.9$$

where μ is the permeability = 4 π/10 μH/metre for air
l is the length of the conductor
A is the cross-sectional area of the conductor.

This equation tells us that the reluctance reduces if the magnetic path is made shorter or is increased in area.
Again notice the similarity between the equations for electrical resistance and the magnetic reluctance. Indeed if the resistivity ρ in Equation 2.8 is replaced by the conductivity σ it becomes $R = (1/\sigma)\, l / A$. If this equation is compared with Equation 2.9 it is seen that the magnetic permeability μ is analogous to the electrical conductivity
If the magnetic circuit is a simple closed magnetic core as in Figure 1 then the meaning of A and l in Equation 2.9 is clear – they are the cross-sectional area of the core and its mean length. But often the magnetic circuit consists of a series of sections of different areas, different lengths and different permeability μ. This is the situation in a coil, where the flux flows down the central hollow section, out of the end and back in at the other end. The magnetic path thus consists of two parts, that inside the coil and that outside, and the total reluctance is the sum of the two.
So for each section we need to add their respective factors $l / \mu A$, and for two sections having A_1, μ_1 and l_1, and A_2 , μ_2 and l_2 respectively we have :

$$\mathcal{R} = [l_1/(\mu_1 A_1) + l_2/(\mu_2 A_1)] \qquad 2.10$$

And the inductance equation 2.7 becomes :

$$\mathbf{L = N^2 / [l_1/(\mu_1 A_1) + l_2/(\mu_2 A_1)] \quad henrys \qquad 2.11}$$

where $\mu = \mu_0 \mu_r$
$\mu_0 = 4\pi/10\ \mu H/m$
μ_r is the relative permeability of the medium (=1 for air)

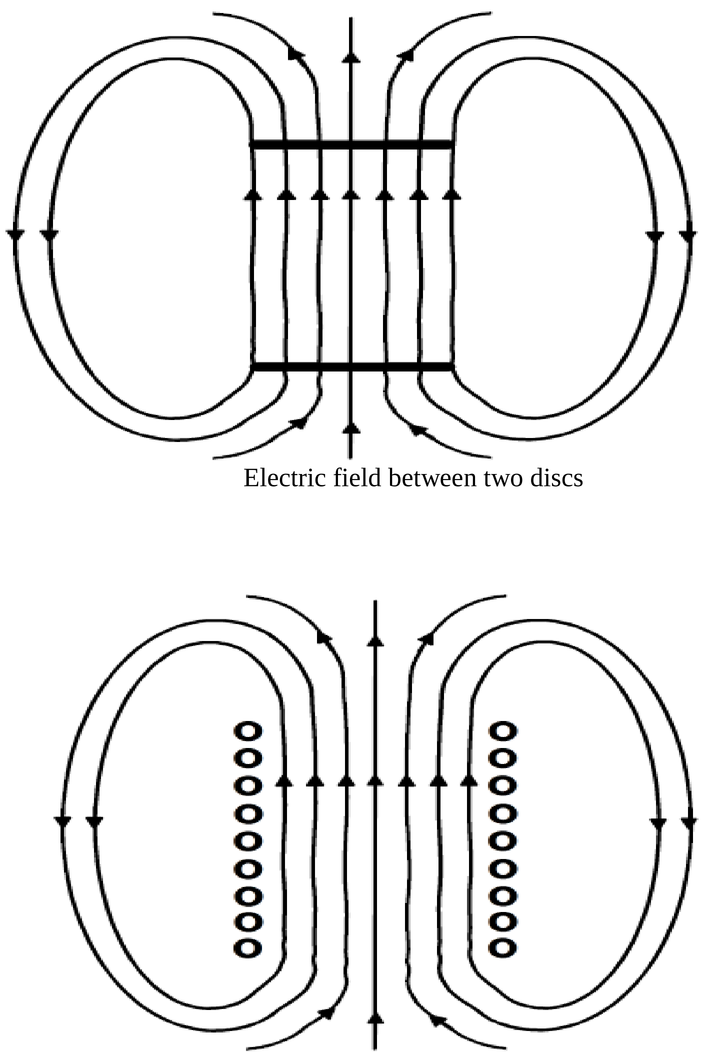
If this is an air coil then the two paths inside the coil and outside are both in air and so $\mu_1 = \mu_2 = \mu_o$, and Equation 2.11 becomes :

$$\mathbf{L = \mu_o N^2 / [l_1/A_1 + l_2/A_1] \quad Henrys \qquad 2.12}$$

The inductance can be calculated therefore if the equivalent length and area of each path, ***l* /A,** is known for each of the magnetic paths, and this is the topic of the next section.
[for an easy-to-read reference on magnetic theory see Snelling ref 2]

3. THE EQUIVALENCE OF CAPACITIVE AND MAGNETIC FIELDS

The previous section showed that to calculate the reluctance it is necessary to know the area and length of the path which the flux takes both inside the coil A_1/l_1 and outside A_2/l_2. There are no known expressions for these magnetic fields but often magnetic fields have the same shape as electric fields, and then the formulae for capacitance can be used to deduce the inductance. Historically more work has been done on electric fields than magnetic, mainly because the capacitance of objects was of such great importance in the early days of radio antennae. So to describe the magnetic field inside and outside a coil, we can look at the equivalent *capacitive* situation which in this case consists of two discs of radius equal to that of the coil, spaced by the length of the coil (see figure 2).



Electric field between two discs

Magnetic field around a coil

Figure 2 Comparison of Electric and Magnetic Fields

The electric field *inside* the discs is described by the familiar expression for a parallel plate capacitor :

$$C_{inside} = \varepsilon_o \, A / l \qquad 3.1$$

where ε_o is the permittivity of free space (ie air)
A is area of the plates
l is the distance between them

However the capacitance of the *external* field from these two discs is more complex and less well known. An approximate equation for the total capacitance of a monopole antenna with capacity hat is provided by Austin (ref 3, p217), which for the dipole consisting of two discs of radius a becomes :

$$C_{mono} \approx \varepsilon_0 \pi a^2/h + 4 \varepsilon_0 a \qquad 3.2$$

where *h* is the distance between the discs
a is the radius of the discs

Notice that the first term is the parallel plate equation given above, accurate on its own when h is very small, and then the second term is negligible in comparison. The second term is due to the external field and is accurate alone when h is very large and then the first term is negligible. So the equation is accurate at the extremes of h but may not be so accurate at intermediate values.
From Equation 3.2 the ratio A/l can be determined for each path, inside and outside. Clearly the first term gives $A_1/l_1 = \pi a^2/h$, or for the coil $A_1/l_1 = \pi a_c^2/lc$ where a_c is the coil radius and lc it length.
But the value of A_2/l_2 in the second term is not so obvious. However, if it is expressed as $4 \varepsilon_0 (h a)/h$ it is seen that the effective area of the external field is (ha), so its area is not constant but increases with height (see also Watt ref 4, p39), or in the case of a coil, increases with its length. Thus $A_2/l_2 = 4a$.
So for a coil :

$$\mathbf{A_1/l_1 = \pi a_c^2/lc} \qquad \mathbf{3.3}$$

$$\mathbf{A_2/l_2 = 4 a_c} \qquad \mathbf{3.4}$$

where a_c is the radius of the coil, and lc its length

These relationships will be used to derive an equation for inductance in the following analysis.

4. DERIVING THE EQUATION FOR INDUCTANCE

It has been shown above that the inductance of an *air* coil can be expressed as (Equation 2.12**)**

$$L = \mu_o N^2/[l_1/A_1 + l_2/A_2] \qquad 4.2$$

where l_1/A_1 is the length and area of the internal flux path
l_2/A_2 is the length and area of the external flux path

The values of A_1/l_1 and A_2/l_2 were derived in Section 3 as Equations 3.3 and 3.4, so:

$$\sum l/A = l_c/(\pi a_c^2) + 1/(4a_c)$$
$$= [4a_c l_c + (\pi a_c^2)]/(4 a_c \pi a_c^2) \qquad 4.3$$

where a_c is the radius of the coil and l_c is its length

Combining 4.2 and 4.3 and putting $\mu_0 = 4\pi/10$ µH/m, gives the inductance of an air coil:

$$L = N^2 4\pi/10 \ \pi a_c^2 / (\pi/4 \ a_c + l_c) \ \mu H \qquad 4.5$$

The accuracy of this equation can be determined by comparing it with the known equations for the inductance of single layer coils, and the most widely used of these is that by Wheeler (see Terman Ref 1, p 55) :

$$Lo = r_i^2 N^2/(9r_i + 10 l_i) \ \mu H \qquad 4.6$$

r_i is the coil radius **in inches**
l_i is the coil length **in inches**
N is the number of turns

Wheeler seems to have derived this equation empirically, but this simple equation is surprisingly accurate 'to within 1% for $l > 0.8a$ ie if the coil is not too short'.
The above equation uses dimensions in inches, whereas this article uses metres. (To avoid any confusion the dimensions in inches have the subscript i). For comparison with Wheeler we put into Equation 4.5, $a_c = r_i$ and $l_c = l_i$, and divide by 39.37 to convert µH/metre to µH/inch, which gives :

$$L_0 = r_i^2 N^2 / (7.86\, r_i + 10\, l_i) \quad \mu H \qquad 4.7$$

Comparing Equations 4.6 and 4.7 there is a small difference in the factor preceding r_i in the denominator. We know that Wheeler's equation is more accurate than Austin's (at least for the magnetic field) so the approximation to the external field derived from Austin's equation above (Equation 3.4) needs to be modified to :

$$A_2/l_2 = 3.49\, a_c \qquad 4.8$$

5. SUMMARY

It is shown that the inductance of a coil can be expressed in terms of the reluctance of its internal and external magnetic paths as:

$$L = \mu_0 N^2 / (\mathcal{R}_{in} + \mathcal{R}_{out}) \quad \text{henrys} \qquad 5.1$$

where $(\mathcal{R}_{in} + \mathcal{R}_{out}) = [l_c/(\mu_{r1}\ \pi\, a_c^{\,2}) + 1/(\mu_{r2}\, 3.49\, a_c)]$

μ_0 is the permeability of free space = 0.4 π µH/m
N is the number of turns
a_c is the coil radius and l_c its length
μ_{r1} is the *relative* permeability of the internal path
μ_{r2} is the *relative* permeability of the external path

The above equation is shown to be identical to Wheeler's equation for the inductance of single layer air coils.

6. FERRITE CORES

Equation 5.1 can be used to determine the inductance when a ferrite core is introduced into the coil. For instance, if a core is inserted of the same diameter and length as the coil, then the reluctance of the *internal* path $l_1/(\mu_{r1} A_1)$ reduces by the relative permeability of the ferrite μ_1, and if this is say 200, the reluctance of this path reduces by 200. However, the inductance of the coil will increase by a much smaller ratio, because the reluctance of the external path has not been affected and this is in series with this low reluctance path. If the ferrite rod is extended beyond the end of the coil the reluctance of the *outside* path $l_2/(\mu_{r2} A_2)$ is also reduced, and the inductance increases further. These aspects are analysed in more detail in a subsequent article (ref 5).

REFERENCES

1. TERMAN F E : Radio Engineers Handbook' First Edition 1943, McGraw-Hill Book Company, Inc.
2. SNELLING E C : 'Soft Ferrites. Properties and Applications' 1969, Illiffe Books London
3. MORECROFT J H 'Principles of Communication' (John Wiley & Sons, 1927)
4. WATT A D : 'VLF Radio Engineering' 1967, Pergamon Press Ltd
5. PAYNE A N : 'The Inductance of Coils containing Ferrite Rods' http://g3rbj.co.uk/

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